L'Hopital's Rules

ONLY USE WHEN YOU HAVE A QUOTIENT!

1) $\pm\infty / \pm\infty$ or $0/0$

2) $0 \cdot (\pm\infty)$, flip one factor to the denominator

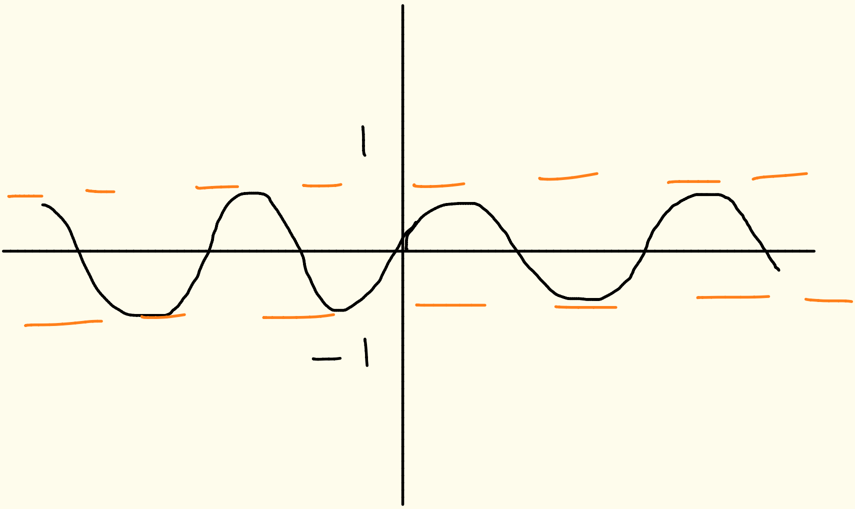
3) $\infty - \infty$, use a common denominator

4) exponent indeterminates, use $e^{\ln}$ trick

Inverse Trig Functions

(Section 6.6)

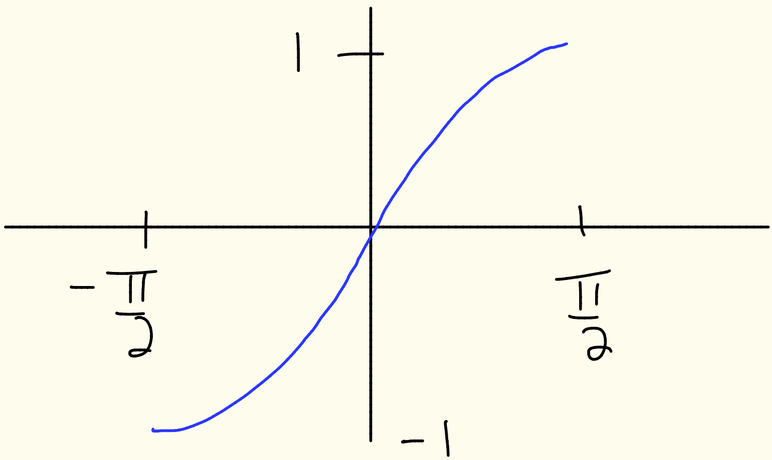
$y = \sin(x)$



is not invertible!

Restriction of domain

Sine isn't invertible, but on the interval from $-\pi/2$ to $\pi/2$, the graph looks like



Restrict sine to $[-\pi/2, \pi/2]$.

The Functions

When $-\pi/2 \leq \theta \leq \pi/2$, denote the inverse function of sine by arcsine.

Domain of arcsine
$=$ range of sine $= [-1, 1]$.

When $-1 \leq x \leq 1$, write $\arcsin(x)$ for the value. In the text or Webwork, you might also see $\sin^{-1}(x)$.

When $-\pi/2 \leq \theta \leq \pi/2$,

$$\boxed{\arcsin(\sin(\theta)) = \theta},$$

and when $-1 \leq x \leq 1$,

$$\boxed{\sin(\arcsin(x)) = x}$$

If $|x| > 1$, $\arcsin(x)$ makes no sense. However, $\sin(\theta)$ makes sense for all θ. When $|\theta| > \frac{\pi}{2}$,

$$\arcsin(\sin(\theta)) \neq \theta \quad .$$

Example 1: Calculate

a) $\sin(\arcsin(-1/2))$

b) $\arcsin\left(\sin\left(\frac{36\pi}{7}\right)\right)$

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a) $-1 \leq -1/2 \leq 1$, so

$$\sin(\arcsin(-1/2)) = \boxed{-\frac{1}{2}}$$

b) $\arcsin\left(\sin\left(\frac{36\pi}{7}\right)\right) =$ whatever number in $[-\pi/2, \pi/2]$ whose sine is equal to $\sin\left(\frac{36\pi}{7}\right)$.

Subtract multiples of 2π (the period of sine) until we get close to our interval.

$$\frac{36\pi}{7} - 2\pi = \frac{36\pi}{7} - \frac{14\pi}{7} = \frac{22\pi}{7} > \frac{\pi}{2}$$

too little

$$\frac{22\pi}{7} - 2\pi = \frac{22\pi}{7} - \frac{14\pi}{7} = \frac{8\pi}{7} > \frac{\pi}{2}$$

too little

$$\frac{8\pi}{7} - 2\pi = \frac{8\pi}{7} - \frac{14\pi}{7} = \frac{-6\pi}{7} < -\frac{\pi}{2}$$

too much!

$\sin\left(\frac{36\pi}{7}\right) = \sin\left(\frac{8\pi}{7}\right).$

$\sin\left(\frac{8\pi}{7} - \pi\right)$ will get

us in the right interval,

but this changes the sine

to negative:

$\arcsin\left(\sin\left(\frac{8\pi}{7}\right)\right)$

$= \arcsin\left(\sin\left(\frac{\pi}{7} + \pi\right)\right)$

$= \arcsin\left(-\sin(\pi/7)\right)$

$= -\arcsin\left(\sin(\pi/7)\right) = \boxed{-\frac{\pi}{7}}$

Since sine an odd function implies arcsine is also odd.

All the other functions

1) arccosine

range: $[0, \pi]$

domain: $[-1, 1]$

$\arccos(\cos(\theta)) = \theta$ when $0 \le \theta \le \pi$

$\cos(\arccos(x)) = x$ when $-1 \le x \le 1$

2) arctangent

range: $(-\pi/2, \pi/2)$

domain: $(-\infty, \infty)$

$\arctan(\tan(\theta)) = \theta$ when $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$

$\tan(\arctan(x)) = x$ always

3) arcsecant

range: $[0, \pi/2) \cup (\frac{\pi}{2}, \pi]$

domain: $(-\infty, 1] \cup [1, \infty)$

$\operatorname{arcsec}(\sec(\theta)) = \theta$ when

$0 \leq \theta < \pi/2$ or $\frac{\pi}{2} < \theta \leq \pi$

$\sec(\operatorname{arcsec}(x)) = x$ when $|x| \geq 1$

4) arccosecant

range: $[-\pi/2, 0) \cup (0, \pi/2]$

domain: $(-\infty, -1] \cup [1, \infty)$

$\operatorname{arccsc}(\csc(\theta)) = \theta$ when

$-\pi/2 \leq \theta < 0$ or $0 < \theta \leq \frac{\pi}{2}$

$\csc(\operatorname{arccsc}(x)) = x$ when $|x| \geq 1$.

5) arccotangent range: $(0, \pi)$

domain: $(-\infty, \infty)$

$\operatorname{arccot}(\cot(\theta)) = \theta$ when $0 < \theta < \pi$

$\cot(\operatorname{arccot}(x)) = x$ always

Note: $\lim_{x \to \infty} \arctan(x) = \frac{\pi}{2}$

$\lim_{x \to -\infty} \arctan(x) = -\frac{\pi}{2}$

Example 2: $\sin(\arctan(12)) = ?$

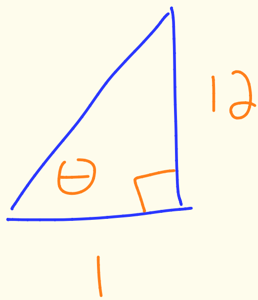
Before we try this:

$\tan(\arctan(12)) = 12$.

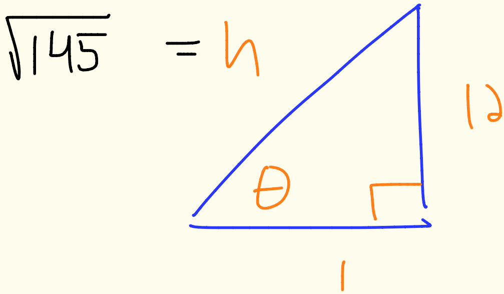
Let $\theta = \arctan(12)$.

We just showed $\tan(\theta) = 12$.

$$\tan(\theta) = \frac{12}{1} = \frac{\text{opp}}{\text{adj}}$$



To find the hypotenuse,
use the Pythagorean Theorem.



$$h^2 = 12^2 + 1^2 = 145, \text{ so } h = \sqrt{145}.$$

$$\sin(\arctan(12)) = \sin(\theta) = \frac{\text{opp}}{\text{hyp}} = \boxed{\frac{12}{\sqrt{145}}}$$

Example 3: $\arctan(\sin(\pi/3))$

$= \arctan\left(\frac{\sqrt{3}}{2}\right)$

$=$ the angle in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ with tangent equal to $\frac{\sqrt{3}}{2}$. I don't know what this number is!

In general, this direction is harder and doesn't work as well.

Inverse Trig Derivatives

Know these

$$\frac{d}{dx}\left(\arcsin(x)\right) = \frac{1}{\sqrt{1-x^2}}$$

$$\boxed{\frac{d}{dx}\left(\arctan(x)\right) = \frac{1}{1+x^2}}$$

$$\frac{d}{dx}\left(\operatorname{arcsec}(x)\right) = \frac{1}{x\sqrt{x^2-1}}$$

These are not as important:

$$\frac{d}{dx}\left(\arccos(x)\right) = \frac{-1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\left(\operatorname{arccot}(x)\right) = \frac{-1}{1+x^2}$$

$$\frac{d}{dx}\left(\operatorname{arccsc}(x)\right) = \frac{-1}{x\sqrt{x^2-1}}$$

All inverse "co" derivatives come with a negative sign, just like regular "co" derivatives.

Example 4: $f(x) = \arccos(\ln(x^2+1))$.

Find $f'(x)$.

$$f'(x) = \frac{-1}{\sqrt{1-(\ln(x^2+1))^2}} \cdot \frac{d}{dx}\left(\ln(x^2+1)\right)$$

$$= \boxed{\frac{-1}{\sqrt{1-(\ln(x^2+1))^2}} \cdot \frac{1}{x^2+1} \cdot 2x}$$

Example 5: Evaluate

$$\int \frac{\frac{dx}{x}}{1+(\ln(x))^2}$$

Substitute $u = \ln(x)$

$$du = \frac{1}{x}\,dx = \frac{dx}{x}$$

The integral becomes

$$\int \frac{du}{1+u^2} = \arctan(u) + C$$

$$= \boxed{\arctan(\ln(x)) + C}$$

Example 6:

Compute

$$\lim_{x \to \infty} \left(\arctan(x) - \frac{\pi}{2}\right) \cdot x$$

$$\lim_{x \to \infty} x = \infty, \quad \lim_{x \to \infty} \left(\arctan(x) - \frac{\pi}{2}\right) = 0$$

Rewrite as a quotient.

$$\lim_{x \to \infty} \frac{\arctan(x) - \pi/2}{\frac{1}{x}}$$

$$\overset{\text{l'H}}{=} \lim_{x \to \infty} \frac{\frac{1}{1+x^2}}{-\frac{1}{x^2}} = \lim_{x \to \infty} \frac{-x^2}{1+x^2}$$

$$\lim_{x \to \infty} \frac{-x^2}{1+x^2} \overset{l'H}{=} \lim_{x \to \infty} \cancel{\frac{-2x}{2x}}$$

$$= \boxed{-1}$$

Chapter 7

Techniques of Integration

Integration by Parts

(Section 7.1)

If you need to know one integration technique, it's this one.

This is just integrating the product rule.

Product Rule

$$(f(x)g(x))' = f'(x)g(x) + g'(x)f(x)$$

$$-f'(x)g(x) \qquad\qquad -f'(x)g(x)$$

we get

$$(f(x)g(x))' - f'(x)g(x) = g'(x)f(x).$$

Integrate both sides with respect to x.

$$\boxed{\int (f(x)g(x))'\,dx} - \int g(x)f'(x)\,dx$$

$$= \int f(x)g'(x)\,dx$$

(fundamental theorem of calculus)

$$\boxed{f(x)g(x)} - \int g(x)f'(x)\,dx = \int f'(x)g(x)\,dx$$

This is integration by parts!